

# LP Duality in Infinite Hypergraphs

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In any graph there exist a fractional cover and a fractional matching satisfying the complementary slackness conditions of linear programming. The proof uses a Gallai–Edmonds decomposition result for infinite graphs. We consider also the same problem for infinite hypergraphs, in particular in the case that the edges of the hypergraph are intervals on the real line. We prove an extension of a theorem of Gallai to the infinite case. © 1990 Academic Press, Inc.

## 1. INTRODUCTION

A *matching* in a hypergraph  $H = (V, E)$  is a set of disjoint edges, and a *cover* of  $H$  is a set of vertices which meets all edges. As usual, we write:  $\nu(H) = \sup\{|F| : F \text{ is a matching in } H\}$  and  $\tau(H) = \inf\{|C| : C \text{ is a cover of } H\}$ .

A *fractional matching* in  $H$  is a function  $f: E \rightarrow [0, 1]$  such that  $\sum_{v \in e} f(e) \leq 1$  for every  $v \in V$ . A *fractional cover* in  $H$  is a function  $g: V \rightarrow [0, 1]$  such that  $\sum_{v \in e} g(v) \geq 1$  for every  $e \in E$ . Given a fractional matching  $f$  and a fractional cover  $g$  there holds

$$\begin{aligned} |f| &:= \sum_{e \in E} f(e) \leq \sum_{e \in E} (f(e) \sum_{v \in e} g(v)) \\ &= \sum_{v \in V} (g(v) \sum_{v \in e} f(e)) \leq \sum_{v \in V} g(v) =: |g|. \end{aligned} \quad (1.1)$$

Hence, defining

$$\nu^* = \sup\{|f| : f \text{ is a fractional matching in } H\}$$

and

$$\tau^* = \inf\{|g| : g \text{ is a fractional cover in } H\}$$

there holds:  $\nu^* \leq \tau^*$ . The duality theorem of linear programming implies that for finite hypergraphs there holds

$$\nu^* = \tau^*. \quad (1.2)$$

Note that (1.1) and (1.2) imply that if  $f$  is a maximal fractional matching (i.e.,  $|f|$  is maximal) and  $g$  a minimal fractional cover, then the following obtains

$$\begin{aligned} (a) \quad & \sum_{v \in e} f(e) = 1 \text{ whenever } g(v) \neq 0, \text{ and} \\ (b) \quad & \sum_{v \in e} g(v) = 1 \text{ whenever } f(e) \neq 0. \end{aligned} \quad (1.3)$$

These are the so-called “complementary slackness conditions.”

Let us now turn to the infinite case. Here (1.2) and (1.3) are no longer equivalent, the latter obviously being a stronger condition. We say that a hypergraph satisfies the *weak duality property* if  $\nu^* = \tau^*$ , and that it has the *strong duality property* if it has a fractional matching  $f$  and a fractional cover  $g$  satisfying (1.3). A hypergraph has the *integral strong duality property* or *König's Property* (KP), if it has a matching  $F$  and a cover  $G$  such that  $G$  consists of the choice of precisely one vertex from each edge in  $F$ . Note that this means that the hypergraph satisfies strong duality with  $f$  and  $g$  being integral (i.e., 0, 1) functions.

The first to suggest the consideration of KP instead of weak duality was Erdős. He conjectured that KP holds for the hypergraph whose edges are the sets of vertices of  $A - B$  paths in a graph, where  $A$  and  $B$  are any sets of vertices. (This is the proper extension of Menger's theorem to the infinite case. It was proved for countable graphs in [3].) It is known that KP holds for bipartite graphs.

In Section 4 we prove that all graphs satisfy the strong duality property, with  $f$  and  $g$  taking 0,  $\frac{1}{2}$ , 1 values. In the finite case this is known as the “2-matching theorem” [6]. This was already proved in [2] for countable graphs. (The original proof of our present result contained a serious error, and we are indebted to K. Steffens for pointing it out to us.) Sections 2 and 3 are devoted to preliminaries to this result. In Section 3 we prove a Gallai-Edmonds type decomposition theorem for general graphs (which is an easy consequence of the results in [4]).

Section 5 is devoted to the proof of KP for families of intervals on the real line with discrete endpoint sets. This is an extension of a theorem of Gallai. In Section 6 we show that König's property fails if accumulation points are allowed, even if the intervals are all of length 1.

In Section 7 we describe the situation in general hypergraphs.

## 2. SOME PRELIMINARIES

In [1] there was proved an extension of König's theorem to infinite bipartite graphs:

**THEOREM 2.1.** *In any bipartite graph there exist a matching  $P$  and a cover  $C$  such that  $C$  consists of the choice of precisely one vertex from each edge in  $P$ .*

If  $\Gamma = (M, W, K)$  is a bipartite graph (the notation means that  $M$  and  $W$  are the sides of  $\Gamma$  and  $K = E(\Gamma)$ ) and  $A \subseteq V(\Gamma) = M \cup W$  and  $a \in V(\Gamma)$ , we write  $K[A]$  for the set of vertices connected to some vertex in  $A$ , and  $K\langle a \rangle$  for  $K[\{a\}]$ . For  $A \subseteq W$  we define  $D(A) = \{x \in M : K\langle x \rangle \subseteq A\}$ .

We say that  $\Gamma$  is *espousable* if there exists in  $\Gamma$  a matching covering  $M$ .

The following is an easy corollary of Theorem 2.1 (in fact, Theorem 2.1 is also easily deducible from it):

**THEOREM 2.2.** *If  $\Gamma$  is inespousable, then there exists  $X \subseteq W$  such that  $X$  is matchable into  $D(X)$  and  $D(X)$  is unmatchable.*

A graph  $D$  is called *matchable* if it contains a perfect matching. It is said to be *factor-critical* if it is unmatchable, but  $D - x$  is matchable for every  $x \in V(D)$ . Given a graph  $G$  and a subset  $S$  of  $V(G)$ , we define a bipartite graph  $\Pi(G, S) = (P, S, K)$  as follows. One side,  $P := P(G, S)$  consists of the factor-critical connected components of  $G - S$ ; the other side is  $S$  itself. The edge set,  $K$ , is defined by  $s \in K\langle Q \rangle$  if  $(s, v) \in E(G)$  for some  $v \in V(Q)$  ( $Q \in P$ ). It is clear that if  $G$  is matchable, then  $\Pi(G, S)$  is espousable for every  $S \subseteq V(G)$ . The converse, which is an infinite version of Tutte's theorem, was proved in [4].

**THEOREM 2.3.** *A graph  $G$  is matchable if and only if  $\Pi(G, S)$  is espousable for every  $S \subseteq V(G)$ .*

Next, we want to prove the strong duality property for factor-critical graphs.

**LEMMA 2.4.** *Let  $Q$  be a factor-critical graph,  $a \in V(Q)$  and  $F$  a perfect matching in  $Q - a$ . Then (i) there does not exist an infinite  $F$ -alternating path starting at  $a$ , and (ii) for every  $x \in V(Q) \setminus \{a\}$  there exists an  $F$ -alternating path from  $a$  to  $x$  terminating at an edge of  $F$ .*

(For the uninitiated, an  $F$ -alternating path is a path in which one of every pair of adjacent edges belongs to  $F$ .)

*Proof.* (i) If  $A$  is an infinite  $F$ -alternating path starting at  $a$ , then  $F \triangle E(A)$  is a perfect matching of  $Q$ , contrary to the assumption that  $Q$  is unmatchable.

(ii) Let  $H$  be a perfect matching of  $Q - x$ , and let  $P$  be the connected component of the graph  $(V(Q), F \cup H)$  containing  $a$ . Then  $P$  is a  $F$ -alternating path, which by part (i) cannot be infinite. But the only way in which it can fail to be infinite is that it terminates at  $x$ , and since  $x \notin \bigcup H$  it follows that the last edge of  $P$  belongs to  $F$ .

**THEOREM 2.5.** *A factor-critical graph with more than one vertex satisfies strong duality, with  $g \equiv \frac{1}{2}$  and  $f$  assuming values of 0,  $\frac{1}{2}$ , and 1.*

*Proof.* Let  $Q$  be a factor-critical graph such that  $|V(Q)| > 1$ . Let  $a \in V(Q)$  and  $F$  be a perfect matching of  $Q - a$ . By part (ii) of the lemma,  $a$  is connected to some vertex  $b$ . (Another way to show this is to note that every factor-critical graph is connected.) Again by the lemma, there exists an  $F$ -alternating path  $R$  from  $a$  to  $b$ , terminating at an edge of  $F$ . Add to  $R$  the edge  $(a, b)$  and denote by  $N$  the resulting circuit. Define:  $f(e) = \frac{1}{2}$  for every  $e \in E(N)$ ;  $f(e) = 1$  for  $e \in F \setminus E(N)$ ;  $f(e) = 0$  for  $e \in E(Q) \setminus (E(N) \cup F)$ ;  $g(v) = \frac{1}{2}$  for every  $v \in V(Q)$ . It is easy to check that  $f$  and  $g$  satisfy conditions (1.3).

The following is an easy extension of the Cantor–Bernstein theorem:

**THEOREM 2.6.** *If  $\Gamma = (M, W, K)$  is bipartite and  $A$  and  $B$  are matchable subsets of  $M$  and  $W$ , respectively, then  $A \cup B$  is matchable.*

### 3. A GALLAI–EDMONDS DECOMPOSITION THEOREM FOR GENERAL GRAPHS

In [8] the results of [5] were used to derive a Gallai–Edmonds type decomposition for countable graphs. Here we shall prove a similar result for general graphs. It is necessarily of weaker form than that of [8].

**THEOREM 3.1.** *In any graph  $G$  the set of vertices  $V$  can be partitioned as  $V = S \cup T \cup R$ , where  $T = \bigcup \{V(P) : P \in \mathbf{P}(G, S)\}$ ,  $S$  is matchable in  $\Pi(G, S)$ ,  $\Pi(G, S)$  is inessential or empty, and  $G[R]$  is matchable.*

(In the countable case one can add the requirement that  $S$  is matchable in  $\Pi(G, S)$  onto a critical subset of  $\mathbf{P}(G, S)$ . For the definition of “critical” see [7]).

*Proof.* We define inductively subsets  $S_\alpha$  of  $V$  ( $\alpha < \theta$  for some ordinal  $\theta < |V|$ ) as follows. Let  $S_0 = \emptyset$ . Assume now that  $S_\beta$  have been defined for

all  $\beta < \alpha$ . Let  $\hat{S}_\alpha = \bigcup_{\beta < \alpha} S_\beta$ ,  $\hat{T}_\alpha = V[\mathbf{P}(G, \hat{S}_\alpha)]$  (if  $\mathbf{F}$  is a family of graphs we denote:  $V[\mathbf{F}] := \bigcup \{V(Q) : Q \in \mathbf{F}\}$ ) and let  $\hat{G}_\alpha = G - \hat{S}_\alpha - \hat{T}_\alpha$ . If  $\hat{G}_\alpha$  is matchable, let  $\theta = \alpha$ , and terminate the process of definition. If  $\hat{G}_\alpha$  is unmatchable then, by Theorem 2.3, there exists a subset  $S^\alpha$  of  $V(G_\alpha)$  such that  $\Pi(\hat{G}_\alpha, S^\alpha)$  is inespousable. By Theorem 2.2 there exists then a subset  $S_\alpha$  of  $S^\alpha$  which is matchable into  $D(S_\alpha)$  in  $\Pi(\hat{G}_\alpha, S^\alpha)$ , and such that  $D(S_\alpha)$  is unmatchable. Clearly  $D(S_\alpha) = \mathbf{P}(\hat{G}_\alpha, S_\alpha)$ , and thus  $S_\alpha$  is matchable in  $\Pi(\hat{G}_\alpha, S_\alpha)$ . Since  $D(S_\alpha)$  is unmatchable it is not empty, and thus  $\hat{T}_\alpha \subsetneq \hat{T}_{\alpha+1}$  (actually, it is also not hard to see that  $S_\alpha \neq \emptyset$ , except for  $\alpha = 0$ ). Hence the process of definition must terminate at some ordinal  $\theta < |V|$ . This happens when  $\hat{G}_\theta$  is matchable (possibly because it is empty). Define then:  $S = \hat{S}_\theta$ ,  $T = \hat{T}_\theta$  and  $R = V \setminus (S \cup T)$ . The required properties follow from the construction.

(*Remark.* Another way to write the same proof is via Zorn's Lemma.)

#### 4. STRONG LP DUALITY FOR GRAPHS

**THEOREM 4.1.** *In any graph  $G = (V, E)$  there exist a fractional matching  $f: E \rightarrow \{0, \frac{1}{2}, 1\}$  and a fractional cover  $g: V \rightarrow \{0, \frac{1}{2}, 1\}$  which satisfy the slackness conditions (1.3).*

*Remark.* In the finite case this is a well-known theorem, see [6].

*Proof.* Let  $V = S \cup T \cup R$  be a partition of  $V$  as in Theorem 3.1. Let  $F$  be a perfect matching of  $G[R]$ . If  $S \cup T = \emptyset$ , let  $f = \chi_F$  (the characteristic function of  $F$ ) and  $g(v) = \frac{1}{2}$  for every  $v \in V$ . Suppose that  $S \cup T \neq \emptyset$ . Let  $\Pi = \Pi(G, S)$ ,  $\mathbf{P} = \mathbf{P}(G, S)$  and let  $X = \{Q \in \mathbf{P} : |V(Q)| > 1\}$ ,  $Y = \{Q \in \mathbf{P} : |V(Q)| = 1\}$ . Consider the graph  $\Pi[S \cup Y]$  and apply to it Theorem 2.1. Let  $P$  and  $C$  be as in that theorem, and let  $A = C \cap Y$ ,  $L = C \cap S$ ,  $B = S \setminus C$ , and  $D = Y \setminus C$ . Let  $H = P \upharpoonright L$  and  $I = P \upharpoonright A$  (note that  $H$  matches  $L$  into  $D$  and  $I$  matches  $A$  into  $B$ ). By the conditions on the partition  $V = S \cup T \cup K$ , we know that  $S$  has a matching  $N$  into  $\mathbf{P}$ . Note that  $N[B] \cap D = \emptyset$ , since  $C$  is a cover in  $\Pi[S \cup Y]$  and  $(B \cup D), C = \emptyset$ . Hence  $N \upharpoonright B$  is a matching of  $B$  in  $\Pi' = \Pi[X \cup A \cup B]$ . The set  $A$  is also matchable in  $\Pi'$ , by the matching  $I$ . Hence, by Theorem 2.6, there exists in  $\Pi'$  a matching  $J$  of  $A \cup B$ . Let  $Z = J[B]$  and for each  $Q \in Z$  choose a vertex  $v = v(Q)$  which is connected to  $J(Q)$  (which belongs to  $B$ ) and let  $K_Q$  be a perfect matching of  $Q - v$ . For every  $Q \in X \setminus Z$  let  $f_Q, g_Q$  be the pair of fractional matching and fractional cover for  $Q$ , whose existence is guaranteed by Theorem 2.5.

We are now ready to define the required fractional matching  $f$  and fractional cover  $g$  for  $G$ . Take a deep breath, and define:  $f(e) = 1$  for every

$e \in F \cup H$  and for every edge of the form  $e = (a, J(a))$ ,  $a \in A$ . Also define  $f((v(Q), J(Q))) = 1$  for every  $Q \in Z$ . For every  $Q \in X \setminus Z$  and  $e \in E(Q)$  let  $f(e) = f_Q(e)$ . Define  $f(e) = 0$  for all other edges.

As for  $g$ , define  $g(v) = 1$  for  $v \in L$ ,  $g(v) = 0$  for  $v \in D$ , and  $g(v) = \frac{1}{2}$  for all other vertices.

The fact that  $f$  is a fractional matching is clear, and it is also not hard to see that  $g$  is a cover: every edge has weight at least  $\frac{1}{2}$  on both its endpoints, unless one endpoint  $x$  is in  $D$ . But, by the definition of  $D$ ,  $x$  is connected only to vertices of  $L$ , on which  $g = 1$ , and hence the edge in question is again covered with weight 1. Condition (a) obtains since  $\sum_{v \in e} f(v) = 1$  for every  $v \in V \setminus D$ , and  $g(v) = 0$  for  $v \in D$ . For condition (b) note that if  $(x, y) \in E$  and  $g(x) + g(y) > 1$ , then at least one of  $x$  or  $y$  belongs to  $L$ . But the only edges with an endpoint in  $L$  on which  $f$  is non-zero, are of the form  $(x, y)$ , where  $x \in L$  and  $y \in D$ , and then  $g(x) + g(y) = 1$ .

## 5. FAMILIES OF INTERVALS ON THE REAL LINE

In this section and the next we consider the duality properties for families of intervals on the real line. A result of Gallai is that finite families of intervals satisfy KP (i.e.,  $\nu = \tau$ ). Here is an extension of this result to the infinite case:

**THEOREM 5.1.** *Any family of intervals on the real line having a discrete set of endpoints satisfies KP.*

For the proof we need a slight extension of Gallai's theorem:

**LEMMA 5.2.** *Let  $H$  be a (possibly infinite) family of closed intervals. If  $\nu(H) = n$  is finite then  $\tau(H) = n$ .*

*Proof of Lemma 5.2.* By induction on  $n$ . For  $n = 0$  there is nothing to prove, so assume  $n > 0$ . Since  $\nu(H)$  is finite, clearly  $t = \sup\{a : [a, b] \in H\}$  is finite. Then  $H' = \{[c, d] \in H : d < t\}$  satisfies  $\nu(H') = n - 1$ , and hence  $H'$  has a cover  $C$  of size  $n - 1$ . Then  $C \cup \{t\}$  is a cover of size  $n$  for  $H$ .

*Proof of Theorem 5.1.* Since the endpoints set is discrete, every interval in  $H$  contains a minimal (by inclusion) member of  $H$ . It is easy to see that the validity of the theorem for the family of all minimal intervals in  $H$  implies it also for  $H$ . Hence we may assume that in  $H$  itself there is no containment relation. For  $A \subseteq H$  and  $x, y \in \mathbb{R}$  write  $A(x, y) = \{I \in A : I \subseteq [x, y]\}$ . Let  $M, M'$  be matchings in  $H$  and  $x, y \in \mathbb{R}$ . Assume

that  $\{x, y\} \cap \bigcup (M \cup M') = \emptyset$  (i.e.,  $x, y$  do not belong to intervals from  $M \cup M'$ ). Write:

$$M|_{[x, y]}^{M'} = (M \setminus M(x, y)) \cup M'(x, y).$$

We say that  $M'$  locally augments  $M$  in  $[x, y]$  by  $m$  if  $|M'(x, y)| = |M(x, y)| + m$ , and that  $M'$  locally augments  $M$  if  $M'$  locally augments  $M$  in  $[x, y]$  by  $m$  for some positive integer  $m$ .

**ASSERTION 5.3.** There exists a matching  $M$  in  $H$  which cannot be locally augmented in any interval  $[x, y]$ .

*Proof of Assertion 5.3.* Let  $M^* = \{[a_k, b_k] : k = \dots, -1, 0, 1, 2, \dots\}$  be a matching in  $H$  constructed as follows. Choose  $[a_0, b_0]$  to be any member of  $H$ . If  $[a_k, b_k]$  is defined and  $k \geq 0$  let  $[a_{k+1}, b_{k+1}]$  be chosen so that  $a_{k+1}$  is minimal satisfying  $a_{k+1} > b_k$  (if such exists). If  $k \leq 0$  let  $[a_{k-1}, b_{k-1}]$  be chosen so that  $b_{k-1}$  is maximal satisfying  $b_{k-1} < a_k$  (if such exists).

Let us first show that any matching  $M \subseteq H$  locally augmenting  $M^*$  in  $[x, y]$  satisfies  $|M(x, y)| = |M^*(x, y)| + 1$ . To see this, we note that for each interval  $I = [a, b] \in M$  with  $a > a_0$  there exists  $[a_k, b_k] \in M^*$  for some  $k = k(I)$  such that  $a_k < a \leq b_k$ , since if  $b_l < a < a_{l+1}$  for some  $l \geq 0$  a contradiction would arise to the choice of  $[a_{l+1}, b_{l+1}]$ .

Define a function  $j_+$  from  $\{[a, b] \in M : a > a_0\}$  into  $\{[a_k, b_k] : k \geq 0\}$  by  $j_+([a, b]) = [a_k, b_k]$  if and only if  $a_k < a \leq b_k$ . Since we assume that no containment exists between intervals in  $H$ , the function  $j_+$  is injective.

A similar argument shows that the map  $j_-$  from  $\{[a, b] \in M : b < b_0\}$  into  $\{[a_k, b_k] : k \leq 0\}$  defined by  $j_-([a, b]) = [a_k, b_k]$  if and only if  $a_k \leq b < b_k$ , is injective.

Then  $j = j_+ \cup j_-$  is a function from  $M$  into  $M^*$  which is injective except possibly for  $[a_0, b_0]$  which can be the image of two members of  $M$ .

We conclude the proof of the assertion by showing that if  $M \subseteq H$  locally augments  $M^*$  in some  $[x, y]$  then  $M$  cannot be locally augmented. For, suppose that  $M'$  locally augments  $M$  in some interval  $[w, z]$ . Then the matching  $(M^*|_{[x, y]}^M)|_{[w, z]}^{M'}$  locally augments  $M^*$  in  $[\min\{x, y\}, \max\{y, z\}]$  by at least 2, contradicting the assertion made above.

The proof of the theorem is complete if we show:

**ASSERTION 5.4.** Let  $H$  be a family of intervals such that  $H(x, y)$  has no infinite matching in any finite interval  $[x, y]$ . A matching  $M \subseteq H$  has a cover of  $H$  consisting of exactly one point from each interval of  $M$  if and only if  $M$  cannot be locally augmented.

*Proof.* Suppose that  $M$  is a matching which cannot be locally augmented. Let  $X = \prod_{I \in M} I$  be taken with the product topology. By Tychonoff's theorem  $X$  is compact. For each  $n > 0$  let

$$u_n = \min\{a_k \geq -n : [a_k, b_k] \in M\}$$

$$v_n = \max\{b_k \leq n : [a_k, b_k] \in M\}$$

and let  $C_n$  be the set of elements  $x$  of  $X$  such that for each  $J \in H(u_n, v_n)$  the  $I$ th coordinate of  $x$  (i.e.,  $x_I$ ), is contained in  $J$  for some  $I \in M(u_n, v_n)$ . Then  $M(u_n, v_n)$  is a finite matching of maximal size in  $H(u_n, v_n)$ . Hence by Lemma 5.2  $C_n \neq \emptyset$ . Since  $C_{n+1} \subseteq C_n$  there holds  $\bigcap_{l=1}^l C_n \neq \emptyset$  for each  $l \geq 1$ . Clearly  $C_n$  is closed and hence  $\bigcap_{n=1}^{\infty} C_n \neq \emptyset$ .

Let  $x$  be any element of  $\bigcap_{n=1}^{\infty} C_n$ . By the definitions of the product space  $X$  and of the covers  $C_n$  the set  $C = \{x_I : I \in M\}$  is a cover of  $H$  consisting of precisely one point from each interval in  $M$ , and the proof of the theorem is complete.

The set of all closed intervals on the real line is a simple counterexample to the theorem when the condition of discreteness of the endpoints set is omitted. In the next section we see that the discreteness condition is necessary even if all intervals in  $H$  are unit intervals, and that even the fractional version of the theorem may fail without this condition.

## 6. A COUNTEREXAMPLE WITH UNIT INTERVALS

The example of all closed intervals in  $\mathbb{R}$  may lead to a question whether a lower bound on the length of the interval suffices for the duality to hold. In this section we see that this is not the case. We present a hypergraph, all of whose edges are unit intervals, which does not satisfy even the fractional duality theorem. In the original version of the paper the proof contained an error, and the present proof is due to Frank Stahlhut from Hannover. We are grateful to him for the correction.

For every  $k \geq 0$  and  $n > 0$  ( $k, n$  natural) let  $I_{k,n} = [k + 1/n, k + 1 + 1/n]$ , and let  $H = \{I_{k,n} : k \geq 0, n > 0\}$ . We show that  $H$  does not satisfy the duality property. Suppose that  $f$  is a fractional matching and  $g$  a fractional cover such that (a) and (b) hold. For every  $k \geq 0, j > 0$  let  $s_j^k = \sum_{n=j}^{\infty} f(I_{k,n})$ . Also write  $s^k := s_1^k$ . Since  $k + 1 \in I_{k,n}$  for every  $n \geq 1$  ( $k \in \mathbb{N}$ ), we have

$$s^k \leq 1. \tag{6.1}$$

Whenever  $s^k > 0$  let  $m_k := \min\{n : f(s_{k,n}) > 0\}$ .



LEMMA 6.1. *For every  $k \geq 0$  there holds:*

- (i)  $s^k = 1$ ;
- (ii)  $g(x) = 0$  for all  $x \in (k, k + 1/m_k)$ ;
- (iii)  $m_k < m_{k-1}$  ( $k > 0$ ).

*Proof.* By induction on  $k$ . Consider first the case  $k=0$ . Assume that  $s^0 < 1$ . Choose  $j$  such that  $s_j' < 1 - s^0$ . It then follows from condition (b) that for all  $x \in (0, 1 + 1/j)$ ,  $g(x) = 0$ , since  $\sum \{f(I): x \in I \in H\} \leq s^0 + s^1 < 1$ . But then  $\sum \{g(x): x \in I_{0,j}\} = 0$ , contradicting the fact that  $g$  is a fractional matching. Part (ii) of the lemma is easy for  $k=0$ , since for  $x \in (0, 1/m_0)$  we have  $\sum \{f(I): x \in I \in H\} < 1$ , by (6.1).

Assume now that the lemma is true for  $k$ . Since  $f(I_k, m_k) > 0$ , we have by condition (a)  $\sum \{g(x): x \in I_{k,m_k}\} = 1$ . Since by the induction hypothesis  $f(x) = 0$  for  $x \in (k, k + 1/m_k)$  and since  $\sum \{g(x): x \in I_{k,n}\} \geq 1$  for all  $n \geq m_k$ , it follows that  $\sum \{g(x): x \in [k + 1/m_k, k + 1]\} = \sum \{g(x): x \in I_{k,m_k}\} = 1$ , and hence

$$g(x) = 0 \quad \text{for all } x \in (k + 1, k + 1 + 1/m_k). \quad (6.2)$$

Assume that  $s^{k+1} < 1$ . Let  $j$  be such that  $s_j^{k+2} < 1 - s^{k+1}$ . Then, for every  $x \in (k + 1 + 1/m_k, k + 2 + 1/j)$  there holds  $\sum \{f(I): x \in I \in H\} \leq s^{k+1} + s_j^{k+2} < 1$ , and hence, by (b),  $g(x) = 0$ . Together with (6.2) this means that  $g(x) = 0$  for every  $x \in (k + 1, k + 2 + 1/j)$ , contradicting the fact that  $g$  is a fractional cover. Thus  $s^{k+1} = 1$ , and (i) is proved for  $k + 1$ .

Since  $f$  is a fractional matching,

$$1 \geq \sum \left\{ f(I): k + 1 + \frac{1}{m_k} \in I \in H \right\} = f(I_{k,m_k}) + s_{m_k}^{k+1},$$

which, since  $f(I_{k,m_k}) > 0$ , implies that  $s_{m_k}^{k+1} < 1$ . By (6.1) this implies that  $m_{k+1} < m_k$ , proving (iii).

For  $x \in (k + 1 + 1/m_k, k + 1 + 1/m_{k+1})$  we have

$$\sum \{f(I): x \in I \in H\} \leq \sum_{n > m_{k+1}} f(I_{k+1,n}) < 1,$$

and hence, by (b),  $g(x) = 0$ . Together with (6.2) this implies that  $g(x) = 0$  for all  $x \in (k + 1, k + 1 + 1/m_{k+1})$ , proving (ii) for  $k + 1$ .

The lemma is thus proved, but of course (iii) is a contradiction, and hence  $H$  does not even satisfy fractional strong duality.

## 7. THE DUALITY PROPERTIES FOR GENERAL HYPERGRAPHS

What happens in general hypergraphs? The following example shows how far are (1.2) and (1.3) from the truth in the general case:

Let  $V = N$  (the set of natural numbers),  $E = \{(k, \infty) : k < \omega\}$ , and  $H = (V, E)$ . Then, clearly,  $\tau(H) = \tau^*(H) = \infty$  and  $\nu(H) = \nu^*(H) = 1$ .

We list here the facts we know about general hypergraphs. The results of this section were obtained in collaboration with Ron Holzman, and the proofs will appear elsewhere.

We say that a hypergraph has *finite character* if its edges are all finite, and that it is *locally finite* if every vertex belongs to finitely many edges.

The following facts are easily proved (some by compactness arguments):

- (1) If  $\tau = \infty$  then  $\tau^* = \infty$  (in fact, if  $\tau = \infty$  then  $\tau^* = \tau$  as cardinalities).
- (2) If  $H$  is of finite character, then  $\tau^*$  is attained (i.e., there exist a fractional cover  $g$  with  $|g| = \tau^*$ ) and  $\nu^* = \tau^*$ .
- (3) If  $H$  is locally finite, then either  $E$  is finite (in which case  $H$  satisfies strong duality) or  $\tau = \infty$ .
- (4) If  $H$  is of finite character and locally finite, then it has the strong duality property.
- (5) If  $\nu^* = \infty$  then there exists a fractional matching  $f$  such that  $|f| = \infty$ .

There exist examples of the following:

- (1) a hypergraph of finite character with  $\nu^* = \tau^*$  finite, in which  $\nu^*$  is not attained (which is equivalent to the failure of the strong duality);
- (2) a locally finite hypergraph in which  $\tau^* = \nu^*$  is finite and  $\nu^*$  is not attained;
- (3) a hypergraph in which  $\tau^*$  is not attained;
- (4) a hypergraph in which  $\nu = 1$  but  $\nu^* = \infty$ .

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